

COUNTING CURVES VIA DEGENERATION

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ABSTRACT. We develop a technique to study curves in a variety which has a degeneration into some union of varieties. The class of such varieties is very broad, but the theory becomes particularly useful when the variety has a degeneration into a union of toric varieties. Hypersurfaces are typical examples, and we study lines on K3 surfaces and quintic Calabi-Yau hypersurfaces in detail. In particular, we combinatorially prove the existence of 2875 lines in a generic quintic Calabi-Yau 3-fold.

1. INTRODUCTION

This paper is, in a sense, a sequel to the paper [11], where we studied higher genus curves in a toric variety. In this paper, we develop a technique which enable us to use ideas from tropical geometry to study curves in a variety which is not necessarily toric. Thus, combining with [11], we can theoretically study higher genus curves in such a variety by combinatorial way.

In [12], we already used the theory of tropical curves to study curves in varieties which degenerate to irreducible toric varieties. Examples included flag varieties and some moduli space of bundles on a Riemann surface. But in this paper, it suffices that the variety has a degeneration to a union of varieties. The class of such varieties is very broad, but we mainly concentrate on the case when the variety has a degeneration to a union of toric varieties ('toric degeneration' in the language of [15]), where the theory becomes particularly efficient.

Our principal examples are K3 and Calabi-Yau hypersurfaces. In the next section, we study the K3 case, where the essential calculation of this paper is done. The core of our calculation consists of two points:

- Calculate the normal sheaf.
- Geometrically understand the tangent and obstruction classes.

The first point is a rather simple calculation, and the result is easily applicable to very broad situations. The second point is a more complicated but interesting calculation. A remarkable point is that the calculation simplifies in (some of) higher dimensional situations. Namely, in Section 3, we study lines in a quintic Calabi-Yau hypersurface. There we borrow an idea Sheldon Katz [6], which we learned from Mark Gross, giving a way to combinatorially count 2875 lines in it. We reprove Katz's counting using our method. In Section 4, we give another example of our calculation, applied to holomorphic disks. Other applications will be given in subsequent papers (see [14]).

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2. COUNTING CURVES ON K3 SURFACES

Consider a degeneration of a quartic K3 surface given by the equation

$$xyzw + tf = 0,$$

where x, y, z, w are homogeneous coordinates of \mathbb{P}^3 , $t \in \mathbb{C}$ is the parameter of the degeneration and f is a generic homogeneous quartic polynomial of x, y, z, w . Let

$$\mathfrak{X} \subset \mathbb{P}^3 \times \mathbb{C}$$

be the variety defined by the above equation (the total space of the degeneration), and X_0 be the central fiber. Let

$$i_0 : X_0 \rightarrow \mathfrak{X}$$

be the inclusion. The space X_0 is the union of four projective planes glued along projective lines:

$$X_0 = \cup_{i=1}^4 \mathbb{P}_i^2$$

Each \mathbb{P}_i^2 has a natural structure of a toric variety, in which the lines mentioned above are the toric divisors. Let

$$\ell_1, \dots, \ell_6$$

be these projective lines. Let $L = \cup_{i=1}^6 \ell_i$ be the union of them. Each ℓ_i has two distinguished points, which are the triple-intersections of the projective planes. We write by ℓ_i° the complement of these two points.

Since f is generic, the total space \mathfrak{X} has twenty four singular points, and for each i , four of them lie on ℓ_i° . Let

$$S \subset X_0$$

be the set of these singular points. These singular points have the same local structure:

Lemma 1. *For each of these singular points, there is an analytic neighborhood isomorphic to a neighborhood of the origin of the set*

$$\{(X, Y, Z, t) \in \mathbb{C}^4 \mid XY + tZ = 0\} \subset \mathbb{C}^3 \times \mathbb{C}.$$

Proof. In fact, in affine coordinates, the equation defining the degeneration becomes

$$xyz + tf = 0.$$

We assume that the line ℓ_i is defined by $x = y = 0$, and one of the singular point on it is given by $z = \alpha$, where $\alpha \neq 0$. Let us write $\tilde{z} = z - \alpha$. Then f can be written in the form

$$f = \tilde{z}f_1 + xf_2 + yf_3,$$

where f_i are polynomials and f_1 has non-zero constant term. Write $\tilde{y} = y(\tilde{z} + \alpha) + tf_2$. The equation becomes

$$x\tilde{y} + t(\tilde{z}f_1 + \tilde{y}f_4 + tf_5),$$

where $f_4 = \frac{f_3}{\tilde{z}+\alpha}$, $f_5 = -\frac{f_2 f_3}{\tilde{z}+\alpha}$. Put $X = x + t f_4$, $Y = \tilde{y}$ and $Z = \tilde{z} f_1 + t f_5$. \square

We consider the following situation. Let

$$\varphi_0 : \mathbb{P}^1 \rightarrow X_0 \subset \mathbb{P}^3$$

be an embedding of a projective line, whose image is contained in an irreducible component of X_0 . The problem is the following:

Determine when φ_0 can be lifted to X_t , $t \neq 0$.

As is pointed out several times ([10], see also [15] in the context of tropical curves), for an immersed stable curve $\psi_0 : C \rightarrow X_0$, if the image is away from the singularities of \mathfrak{X} , it is necessary for ψ_0 to satisfy the *pre-log* condition (Definition 4.3, [15]) to solve the above problem. In our situation, the line is embedded in a component of X_0 , so we have the following necessary condition.

Lemma 2. *Let φ_0 be as above. Let \mathbb{P}_i^2 be the component of X_0 to which the line is mapped. Then, to solve the problem above, it is necessary to satisfy the condition:*

$$(*) \quad \varphi_0(\mathbb{P}^1) \cap L \subset S.$$

\square

Since the set S is contained in $\cup_{i=1}^6 \ell_i^\circ$, one sees that φ_0 is torically transverse as a map to \mathbb{P}_i^2 , in the sense of Definition 4.1 of [15].

Remark 3. *For the condition $(*)$ to be satisfied, a condition must be imposed to f , the defining polynomial of the quartic surface. Namely, the set $S \cap \mathbb{P}_i^2$ should contain three points which are collinear. This imposes one dimensional condition to f , and corresponds to the fact that generic K3 surface does not contain an embedded (-2) -curve. In the rest of this section, we take generic f among those which satisfy this condition. In particular, we still assume that the singular locus S does not intersect the torus fixed point set of \mathbb{P}_i^2 .*

The condition $(*)$ is necessary, but not sufficient. For example, consider the family defined by the equation

$$xyzw + t(x^4 - z^4 - 2zw^3 - w^4 + yw^3 + y^4) = 0$$

using homogeneous coordinates. The set S contains the points

$$(x, y, z, w) = (1, 0, 0, 1), \quad (0, 0, -1, 1), \quad (1, 0, 1, 0)$$

so the line

$$x - z - w = 0, \quad y = 0$$

satisfies the condition $(*)$. However, by direct calculation, one sees that there is no first order lift of this line. The rest of this section is devoted to the study of the obstruction to the existence of lifts of lines satisfying $(*)$.

2.1. Calculation of the cohomology class. As usual in deformation theory, the possible obstructions are represented by some cohomology classes. However, whether these obstructions really matter or not can not be seen just by looking at the cohomology classes. This is a non-linear problem, and solved by calculating *Kuranishi maps*. We calculate the cohomology classes in this subsection, and calculate Kuranishi maps in the next subsection.

2.1.1. Identification of the normal sheaf. First we study the local nature around the singular points. As noted above, the space \mathfrak{X} has an neighborhood \mathfrak{U} which is analytically isomorphic to the variety defined by the equation $XY + tZ = 0$ at each singular point. We consider an embedded line in $\mathbb{P}_i^2 \subset X_0$ which

- is torically transverse as a map to \mathbb{P}_i^2 and
- intersects the toric boundary at the singular points,

as above.

From the proof of Lemma 1, we see that locally the line is given by the equations

$$aX + bZ + ZT(X, Z) = 0, \quad Y = t = 0,$$

where a, b are generic complex numbers. In particular, neither of a, b is zero. $T(X, Z)$ is a convergent series around $X = Z = 0$ whose constant term is zero. We take an affine coordinate S on \mathbb{P}^1 so that

$$\varphi_0^* X = S.$$

Here we write the composition

$$i_0 \circ \varphi_0 : \mathbb{P}^1 \rightarrow X_0 \rightarrow \mathfrak{X}$$

by the letter φ_0 , for brevity. Now we study how the local lifts of φ_0 are described using these coordinates.

Note that the variety $XY + tZ = 0$ has a natural structure of a toric variety over $\text{Spec } \mathbb{C}[t]$. So it has a standard log structure which is log smooth over $\text{Spec } \mathbb{C}[t]$ ($\text{Spec } \mathbb{C}[t]$ is also equipped with the standard log structure as a toric variety). Also, we put a log structure on \mathbb{P}^1 associated to the divisor $z_0 = \{S = 0\}$.

The logarithmic tangent sheaf $\Theta_{\mathfrak{U}}$ of \mathfrak{U} is locally free and generated by the sections

$$X\partial_X, \quad Y\partial_Y, \quad Z\partial_Z.$$

There is a following relation between the logarithmic cotangent vectors:

$$\frac{dX}{X} + \frac{dY}{Y} - \frac{dZ}{Z} - \frac{dt}{t} = 0,$$

which becomes

$$\frac{dX}{X} + \frac{dY}{Y} - \frac{dZ}{Z} = 0$$

when restricted to X_0 .

Since the map φ_0 factors through X_0 , the natural map from $\Theta_{\mathbb{P}^1}$ to $\varphi_0^* \Theta_{\mathfrak{X}}$ is locally given by

$$S\partial_S \mapsto (X\partial_X - Y\partial_Y) + (1 + \tilde{T}(X, Z))(Z\partial_Z + Y\partial_Y)$$

around z_0 . Here $\tilde{T}(X, Z)$ is a convergent series around $X = Z = 0$ whose constant term is zero.

Since $\mathfrak{X} \rightarrow \mathbb{C}$ is log smooth, there is always a local lift of φ_0 . The obstruction to the existence of a lift of φ_0 is given by the first cohomology of the pull-back $\varphi_0^*\Theta_{\mathfrak{X}}$ ([7]). If a lift exists, the space of such lifts is a torsor of the zeroth cohomology of the same sheaf. However, since the domain \mathbb{P}^1 (with three special points) has no moduli, it suffices to replace $\varphi_0^*\Theta_{\mathfrak{X}}$ by the logarithmic normal sheaf for the calculation of the obstruction.

Around z_0 , the logarithmic normal sheaf $\mathcal{N}_{\mathfrak{X}/\mathbb{P}^1}$ of $\varphi_0(\mathbb{P}^1)$ is the quotient of $\varphi_0^*\Theta_{\mathfrak{X}}$ by the subsheaf generated by the section $(X\partial_X - Y\partial_Y) + (1 + \tilde{T}(X, Z))(Z\partial_Z + Y\partial_Y)$ above, so it is also locally free and generated by $Y\partial_Y, X\partial_X - Y\partial_Y$.

Since we consider a lift of φ_0 over $\text{Spec } \mathbb{C}[t]$, the normal vector which gives a local lift must be evaluated to one by the covector $\frac{dt}{t}$. So the local lifts are given by the local sections

$$(\bullet) \quad Y\partial_Y + g(X\partial_X - Y\partial_Y)$$

of $\varphi_0^*\mathcal{N}_{\mathfrak{X}/\mathbb{P}^1}$, where g is a holomorphic function on a neighborhood of $z_0 \in \mathbb{P}^1$.

Let

$$j_0 : \mathbb{P}_i^2 \rightarrow \mathfrak{X}$$

be the inclusion. Around $\varphi_0(z_0)$, we induce a log structure on \mathbb{P}_i^2 from that of \mathfrak{X} . There is a natural inclusion from $\Theta_{\mathbb{P}_i^2}$ (which is spanned by $X\partial_X - Y\partial_Y$ and $Z\partial_Z + Y\partial_Y$ around $\varphi_0(z_0)$) to $j_0^*\Theta_{\mathfrak{X}}$, and this again factors through X_0 . But we remark that the coordinates X, Z are not the standard coordinates of an affine part of \mathbb{P}_i^2 , which we wrote by x, z . Using $\tilde{z} = z - \alpha$ instead of z as in the proof of Lemma 1, the relation between logarithmic tangent vectors of the two coordinate systems are given by

$$x\partial_x \mapsto (X\partial_X - Y\partial_Y) + \frac{x\partial_x f_1}{f_1}(Z\partial_Z + Y\partial_Y), \quad \tilde{z}\partial_{\tilde{z}} \mapsto \left(1 + \frac{\tilde{z}\partial_{\tilde{z}} f_1}{f_1}\right)(Z\partial_Z + Y\partial_Y).$$

The map from $\Theta_{\mathbb{P}^1}$ to $\varphi_0^*\Theta_{\mathfrak{X}}$ factors through this. In the coordinate x, \tilde{z} , the line is given by the equation

$$ax + b\tilde{z} = 0$$

as a subvariety of \mathbb{P}_i^2 . The map from $\Theta_{\mathbb{P}^1}$ to $\varphi_0^*\Theta_{\mathbb{P}_i^2}$ is given by

$$S\partial_S \mapsto x\partial_x + \tilde{z}\partial_{\tilde{z}}.$$

(here again we abused the notation by representing the map $\mathbb{P}^1 \rightarrow \mathbb{P}_i^2 \subset X_0$ by the same letter φ_0).

So around z_0 , the subsheaf of $\varphi_0^*\mathcal{N}_{\mathfrak{X}/\mathbb{P}^1}$ generated by $X\partial_X - Y\partial_Y$ is naturally isomorphic to the quotient $\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1}$ of $\varphi_0^*\Theta_{\mathbb{P}_i^2}$ by the subsheaf generated by $x\partial_x + \tilde{z}\partial_{\tilde{z}}$:

$$\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1} \langle x\partial_x, \tilde{z}\partial_{\tilde{z}} \rangle / x\partial_x + \tilde{z}\partial_{\tilde{z}}.$$

On the other hand, the usual (non-logarithmic) normal bundle of the line $ax + b\tilde{z}$ in \mathbb{P}^2 is, locally around z_0 , given by the quotient

$$\mathcal{Q} = \mathcal{O}_{\mathbb{P}^1} \langle \partial_x, \partial_{\tilde{z}} \rangle / b\partial_x - a\partial_{\tilde{z}}.$$

There is a natural map from $\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1}$ to \mathcal{Q} (note that since $ax + b\tilde{z} = 0$, $x\partial_x + \tilde{z}\partial_{\tilde{z}} = \frac{x}{b}(b\partial_x - a\partial_{\tilde{z}})$).

This is isomorphic except $S = 0$, and at $S = 0$, the quotient is the skyscraper sheaf $\mathcal{O}_{S=0}$. It is clear that away from the singularity, the logarithmic normal bundle $\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1}$

is naturally isomorphic to the usual normal bundle. Since there are three singular points which are locally analytically isomorphic, we have the exact sequence of sheaves

$$0 \rightarrow \mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1} \rightarrow \mathcal{Q} \rightarrow \mathcal{O}_{p_1} \oplus \mathcal{O}_{p_2} \oplus \mathcal{O}_{p_3} \rightarrow 0,$$

here p_i are the points on \mathbb{P}^1 mapped to S by φ_0 . The usual normal bundle \mathcal{Q} is isomorphic to $\mathcal{O}_{\mathbb{P}^1(1)}$, so the sheaf $\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-2)$.

As seen from (\bullet) , this sheaf $\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1}$ plays the role of the normal sheaf for $\varphi_0(\mathbb{P}^1)$ (over $\text{Spec } \mathbb{C}[t]$).

2.1.2. *Calculation of the (dual) obstruction class.* The cohomology groups of $\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1}$ are

$$H^0(\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1}) = 0, \quad H^1(\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1}) = \mathbb{C}.$$

That is, there is an obstruction class for φ_0 . For the calculation of the Kuranishi map, we need to represent this class in an effective way.

By construction, $\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1}$ is the sheaf of sections of the usual normal sheaf \mathcal{Q} which have single zeroes at three points. By Serre duality,

$$H^1(\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1}) \cong (H^0((\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1})^\vee \otimes \omega_{\mathbb{P}^1}))^\vee.$$

By the above description, $(\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1})^\vee$ is the sheaf of sections of the usual conormal bundle \mathcal{Q}^\vee , where single poles are allowed at three points.

From the point of view of tropical geometry, the line is presented by the 1-skeleton of the standard fan on $N_{\mathbb{R}} = \mathbb{R}^2$ defining \mathbb{P}^2 up to translation. The directions of the three edges from the unique vertex are

$$(1, 0), \quad (0, 1), \quad (-1, -1).$$

In particular, the sum of these vectors is zero, by the balancing condition (see [11] for the basic notions of tropical curves). These edges correspond to the intersection of the line with the toric divisors. So these edges also corresponds to the intersection of the line with the singular locus S , where the sections of $(\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1})^\vee$ can have poles.

Then it is easy to see that the sections of $(\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1})^\vee$ is represented by the triplet of vectors in the dual space $M_{\mathbb{R}}$ of $N_{\mathbb{R}}$, which annihilate the vectors in the directions of the three edges, respectively. That is, taking $e_1 = (1, 0)$ and $e_2 = (0, 1)$ as a basis of $N_{\mathbb{R}}$ and writing by f_1, f_2 the dual basis of $M_{\mathbb{R}}$, we have

$$H^0(\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1})^\vee \cong \mathbb{C}\langle f_1 \rangle \oplus \mathbb{C}\langle -f_2 \rangle \oplus \mathbb{C}\langle -f_1 + f_2 \rangle.$$

The sections of the normal sheaf \mathcal{Q} of the line is identified with the vectors of $N_{\mathbb{R}}$. So they make natural parings with the vectors f_1, f_2 and $f_1 - f_2$. The paring between a section of $(\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1})^\vee \otimes \omega_{\mathbb{P}^1}$ and a section of \mathcal{Q} gives a rational section of the canonical bundle $\omega_{\mathbb{P}^1}$, and its residues at the poles is given by the value of the paring between an element of $\mathbb{C}\langle f_1 \rangle \oplus \mathbb{C}\langle -f_2 \rangle \oplus \mathbb{C}\langle -f_1 + f_2 \rangle$ and the vector of $B_{\mathbb{R}}$ corresponding to the section of \mathcal{Q} .

By residue theorem, the sum of the residues must be zero. In terms of the vectors, this means that the sections of $(\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1})^\vee \otimes \omega_{\mathbb{P}^1}$ is represented by the subspace of $\mathbb{C}\langle f_1 \rangle \oplus \mathbb{C}\langle -f_2 \rangle \oplus \mathbb{C}\langle -f_1 + f_2 \rangle$ as follows:

$$H^0((\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1})^\vee \otimes \omega_{\mathbb{P}^1}) \cong \{(a, b, c) \in \mathbb{C} \mid af_1 + b(-f_2) + c(-f_1 + f_2) = 0\} = \mathbb{C} \cdot (1, 1, 1).$$

Summarizing, we have the following.

Theorem 4. *Let $\varphi_0 : \mathbb{P}^1 \rightarrow X_0 \subset \mathfrak{X}$ be a line satisfying the condition (*) of Lemma 2. The cohomology group representing the obstruction to the existence of a lift of φ_0 is isomorphic to \mathbb{C} , and its dual space is naturally presented by the triplet of vectors*

$$v_1 \in \mathbb{C} \cdot f_1, \quad v_2 \in \mathbb{C} \cdot f_2, \quad v_3 \in \mathbb{C} \cdot (f_1 - f_2)$$

in $M_{\mathbb{R}}$ satisfying the condition

$$v_1 + v_2 + v_3 = 0.$$

□

2.2. Calculation of the Kuranishi map. The next task is to see whether this class really obstructs the existence of a lift of φ_0 or not. We begin with studying the local model around the singularity. Namely, consider

$$XY + tZ = 0$$

and a curve C given by a parametrization:

$$X = s, \quad Z = s\zeta(s), \quad Y = t = 0.$$

Here ζ is an analytic function and s is a coordinate on the curve C . Note that this curve intersects the singular point

$$(X, Y, Z, t) = (0, 0, 0, 0).$$

We study a general form of lifts of C (over $\mathbb{C}[t]$). They should have the form

$$\begin{aligned} X &= s + tf_1(s) + t^2f_2(s) + \cdots, \\ Z &= s\zeta(s) + tg_1(s) + t^2g_2(s) + \cdots, \\ Y &= th_1(s) + t^2h_2(s) + \cdots, \end{aligned}$$

where f_i, g_i, h_i are analytic functions.

Taking $S = s + tf_1(s) + t^2f_2(s) + \cdots$ as a new coordinate on C , the above parametrization for X, Y, Z becomes

$$\begin{aligned} X &= S, \\ Z &= (S - tf_1)\zeta(S - tf_1) + tg_1(S - tf_1) + \cdots = S\zeta(S) + t\bar{g}_1(S) + t^2\bar{g}_2(S) + \cdots, \\ Y &= t\bar{h}_1(S) + t^2\bar{h}_2(S) + \cdots, \end{aligned}$$

where \bar{g}_i, \bar{h}_i are analytic functions. By $XY + tZ = 0$,

$$\bar{h}_1(S) = -\zeta(S), \quad S\bar{h}_{i+1}(S) + \bar{g}_i(S) = 0, \quad i \geq 1.$$

Thus, using appropriate coordinate on C , we have the expression of a general lift as

$$\begin{aligned} X &= S, \\ Z &= -Sh_1(S) - tSh_2(S) - t^2Sh_3(S) - \cdots, \\ Y &= th_1(S) + t^2h_2(S) + \cdots. \end{aligned}$$

In particular, for any order of t , the lift intersects $X = Z = 0$ at the parameter $s = 0$. This can also be seen from the discussion in the previous subsection, since the section of the normal sheaf must have a single zero at each singular point, compared to the 'usual' normal sheaf of a line in \mathbb{P}^2 .

Now we return to the realistic situation, where we consider a degeneration of K3 surfaces.

Recall that in the proof of Lemma 1, we introduced a change of coordinates

$$X = x + tf_4, \quad Y = \tilde{y}, \quad Z = \tilde{z}f_1 + tf_5$$

to locally bring the equation into the model case $XY + tZ = 0$.

In terms of the coordinates x, y, \tilde{z} , the line in X_0 is represented as

$$x = s, \quad \tilde{z} = as, \quad t = y = 0,$$

where a is a nonzero complex number. Lifts of this must have the form (after appropriate affine change of the coordinates on the domain curve)

$$\begin{aligned} x &= s, \\ \tilde{z} &= as + t(a_1s + b_1) + t^2(a_2s + b_2) + \cdots, \\ y &= t(c_1s + d_1) + t^2(c_2s + d_2) + \cdots. \end{aligned}$$

The relation between the parameters s and S of the line (near a singular point) is given by

$$S = X = s + tf_4(s, t).$$

Substituting this to Z , we have

$$\begin{aligned} \tilde{z}f_1 + tf_5 &= (as + t(a_1s + b_1) + t^2(a_2s + b_2) + \cdots)f_1(s, t) + tf_5(s, t) \\ &= -Sh_1(S) - tSh_2(S) - t^2Sh_3(S) - \cdots \\ &= -(s + tf_4(s, t))h_1(s + tf_4(s, t)) - t(s + tf_4(s, t))h_2(s + tf_4(s, t)) - \cdots. \end{aligned}$$

Note that $f_1(s, t)$, $f_4(s, t)$ and $f_5(s, t)$ contains the variable t because these are functions of x, \tilde{z}, y , and \tilde{z} and y have the above form.

Comparing the coefficients of s^i , $i = 1, 2, \dots$, we see

$$asf_1(s, 0) = -sh_1(s).$$

So we have

$$h_1(s) = -af_1(s, 0).$$

This determines h_1 .

Now let us compare the coefficients of t . We have

$$b_1f_1(0) + f_5(0) = -f_4(0)h_1(0).$$

Since we know h_1 , we can calculate b_1 from this (recall $f_1(0) \neq 0$). Note that although the choices of f_2 and f_3 are not unique, but one sees that b_1 does not depend on this choice.

Remark 5. *Observe that in the model case $XY + tZ = 0$, all lifts of a curve have the form*

$$Z = -Sh_1(S) - tSh_2(S) - \cdots,$$

that is, $b_i = 0$ for all i . In particular, the nonzero b_i is caused by global reason, and they are the obstruction to the lift.

The Kuranishi map is calculated as follows. Recall that the dual of the obstruction class was described by conormal sheaf valued 1-forms where a logarithmic pole is allowed at each singular point, satisfying some condition. In turn, these sections are described by a tripled of vectors in $M_{\mathbb{R}}$, the dual space of $N_{\mathbb{R}}$ where the fan defining \mathbb{P}_i^2 lies. The term b_1 corresponds to a normal vector $b_1 \frac{\partial}{\partial z} = \frac{b_1}{\alpha} \cdot z \frac{\partial}{\partial z}$ at the singular point (recall α is the value of z at the singular point), and this gives a vector in $N_{\mathbb{R}}$ (more precisely, a vector in the quotient space of it by the subspace which the kernel of a vector in $M_{\mathbb{R}}$ describing the

dual obstruction). At each of the three singular points, we have similarly such a vector, and making pairings between these vectors and the above three vectors in $M_{\mathbb{R}}$ and taking the sum, we get a scalar. This is just the Kuranishi map.

Remark 6. *Since the zeroth cohomology of the normal sheaf $\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1}$ is zero and the first cohomology is one dimensional, the Kuranishi map can be represented as a map from a point to \mathbb{C} , that is, it is given by a scalar.*

This is the first order Kuranishi map. When this is zero, then we can solve the equations for a_1, c_1 and d_1 and determine the first order lift. Then doing the same procedure for b_2 , we can define the second order Kuranishi map, and when this is zero, we can find the second order lift, and so on.

2.3. Example. As an example of our calculation, we consider a degeneration given by an equation of the following form:

$$xyzw + t(x^4 - z^4 - 2zw^3 - w^4 + ax^2w^2 + bxzw^2 + cz^2w^2 + dxw^3 + yw^3 + y^4) = 0.$$

Here a, b, c, d are unfixed scalars. This equation is chosen in the following way: Consider a simpler equation

$$xyzw + t(x^4 - z^4 - 2zw^3 - w^4 + yw^3 + y^4) = 0.$$

This is the equation we gave after Remark 3. The singular locus of this degeneration contains three points

$$(x, y, z, w) = (1, 0, 0, 1), (0, 0, -1, 1), (1, 0, 1, 0),$$

which lie on the line

$$x - z - w = 0, \quad y = t = 0.$$

However, it is easy to see (by calculating the Kuranishi map, or by calculating directly) that this line does not have the first order lift. So we add terms $ax^2w^2 + bxzw^2 + cz^2w^2 + dxw^3$, and study when the above line has the first order lift.

First of all, the singular locus of the total space \mathfrak{X} must contain the above three points. This implies the condition

$$a + d = 0, \quad c = 0.$$

So the equation becomes

$$xyzw + t(x^4 - z^4 - 2zw^3 - w^4 + ax^2w^2 + bxzw^2 - axw^3 + yw^3 + y^4) = 0.$$

Now we assume $w \neq 0$ and inhomogenize the equation:

$$xyz + t(x^4 - z^4 - 2z - 1 + ax^2 + bxz - ax + y + y^4) = 0.$$

Consider the singular point $(x, y, z) = (1, 0, 0)$. The parametrization is

$$x = s + 1, \quad z = s, \quad y = 0.$$

The functions f_i are given as follows:

$$f_1 = x^3 + x^2 + (1 + a)x + 1,$$

$$f_2 = -z^3 - 2 + bx,$$

$$f_3 = 1 + y^3.$$

Then

$$f_4 = \frac{1+y^3}{x}, \quad f_5 = \frac{(1+y^3)(z^3+2-bx)}{x}.$$

Since

$$b_1 = \frac{f_1(0)f_4(0) - f_5(0)}{f_1(0)},$$

we have

$$b_1 = \frac{4+a-2+b}{4+a} = \frac{a+b+2}{4+a}.$$

Similarly, at $(x, y, z) = (0, 0, -1)$, one calculates

$$b_1 = \frac{a+b-2}{2}.$$

At the remaining singularity which lies at $x, z \rightarrow \infty$, one sees $b_1 = 0$. Thus, the Kuranishi map is given by

$$\frac{a+b+2}{4+a} + \frac{a+b-2}{2}.$$

The condition for the vanishing of the Kuranishi map is

$$a^2 + ab + 4a + 6b - 4 = 0.$$

On the other hand, we can directly calculate the liftability condition by substituting

$$x = s,$$

$$\tilde{z} = as + t(a_1s + b_1) + t^2(a_2s + b_2) + \cdots,$$

$$y = t(c_1s + d_1) + t^2(c_2s + d_2) + \cdots.$$

into the equation. Then from the coefficients of ts^i , we have

$$c_1 = -4, \quad d_1 = 2 - a - b.$$

Using this, we see from the coefficients of t^2s^i ,

$$a_1 = 0, \quad 2b_1 + 2 - a - b = 0, \quad (a+6)b_1 + 4 = 0.$$

Solving this, we again have

$$a^2 + ab + 4a + 6b - 4 = 0.$$

In general, one easily sees that the Kuranishi map drastically reduces the amount of calculation.

Remark 7. *Each component of X_0 is \mathbb{P}^2 with four distinguished points (singular locus of \mathfrak{X}) on each component of the toric divisors. We considered a line through three of these points, then there is a unique cubic curve through the remaining nine points. If the line is liftable, then the intersection of the hyperplane containing the lifted line with the K3 surface contains another cubic curve, so the above calculation of the Kuranishi map also calculates the Kuranishi map for the cubic curve.*

2.4. Cubic surface. As an easy application of the calculation of the proof of Theorem 4, we consider the case of a cubic surface. We consider a degeneration similar to the K3 case:

$$xyz + tf = 0,$$

where x, y, z are three of the standard homogeneous coordinates of \mathbb{P}^3 , $t \in \mathbb{C}$ is the parameter for the degeneration, and f is a generic homogeneous cubic polynomial. Let $\mathfrak{X} \subset \mathbb{P}^3 \times \mathbb{C}$ be the total space of the degeneration, and $X_0 \subset \mathfrak{X}$ be the central fiber as before. In this case, X_0 is a union of three projective planes glued along three projective lines:

$$X_0 = \mathbb{P}_1^2 \cup \mathbb{P}_2^2 \cup \mathbb{P}_3^2,$$

$$\mathbb{P}_i^2 \cap \mathbb{P}_{i+1}^2 = \ell_i, \quad (i \in \mathbb{Z}/3\mathbb{Z}).$$

On each of these lines, there are three singular points of \mathfrak{X} . See the figure below.

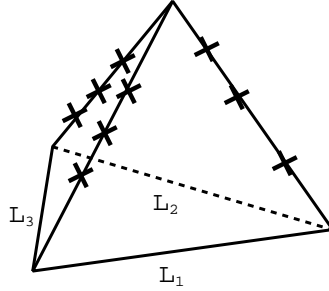


FIGURE 1.

Note that the bottom is open. Each singular point has the same local structure as in the K3 case.

We consider the problem of which line in X_0 lifts to a general fiber of \mathfrak{X} . So let

$$\varphi_0 : \mathbb{P}^1 \rightarrow \mathbb{P}_i^2$$

be a line. As in the K3 case, if φ_0 is liftable, then it is necessary that the intersection with ℓ_i must be at the singular points. However, in this case, there is no restriction to the intersection with the bottom lines (L_1, L_2, L_3 in the figure).

The relevant normal bundle $\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1}$ is

$$\mathcal{O}_{\mathbb{P}^1}(1 - 2) = \mathcal{O}_{\mathbb{P}^1}(-1)$$

by the calculation in the previous proof. In this case, the cohomology groups are

$$H^0(\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1}) = 0, \quad H^1(\mathcal{N}_{\mathbb{P}_i^2/\mathbb{P}^1}) = 0.$$

In particular, the line is unobstructed. Thus, any line which intersects ℓ_i at singular points lifts, and on each \mathbb{P}_i^2 , there are $3 \times 3 = 9$ such lines. So there are $9 \times 3 = 27$ liftable lines in total, which proves the well-known result for the cubic surface.

3. COUNTING LINES IN QUINTIC CALABI-YAU HYPERSURFACES

3.1. Combinatorics of lines in the central fiber. Here we review an idea to combinatorially count (-1) -curves in quintic Calabi-Yau hypersurfaces via degeneration, due to Sheldon Katz [6].

Consider a degeneration of a generic quintic Calabi-Yau hypersurface defined by the equation

$$z_0 \cdots z_4 + tf = 0,$$

where z_0, \dots, z_4 are homogeneous coordinates of \mathbb{P}^4 , $t \in \mathbb{C}$ is the parameter of degeneration and f is a generic homogeneous quintic polynomial of z_0, \dots, z_4 . Let \mathfrak{X} be the total space and

$$X_0 = \bigcup_{i=1}^5 \mathbb{P}_i^3$$

be the central fiber. Each \mathbb{P}_i^3 has a natural structure of a toric variety. Let us consider \mathbb{P}_1^3 . In this case, the intersection of \mathbb{P}_1^3 with the singular locus S of \mathfrak{X} is given by the union of four quintic curves, one for each toric divisor:

$$\mathbb{P}_1^3 \cap S = \bigcup_{j=2}^5 C_j, \quad C_j \subset \mathbb{P}_1^3 \cap \mathbb{P}_j^3$$

Moreover, for $j_1 \neq j_2$, $C_{j_1} \cap C_{j_2}$ is the set of five points. Since f is generic, we can assume that C_j does not intersect the torus fixed points of \mathbb{P}^1 .

We consider the lifting problem (P) for lines in \mathbb{P}_1^3 . As in the K3 case, a liftable line should intersect the toric divisor of \mathbb{P}_1^3 only at $\mathbb{P}_1^3 \cap S$. Since a line intersects each toric divisor only once and C_j s lie on the toric boundary, this condition is the same as the condition that the line intersects each C_j . Since the degree of $Gr(2, 4)$ is two, there are at least

$$2 \cdot 5^4$$

lines satisfying this condition, for generic f .

Remark 8. *Even if f is generic, the configuration of the four quintic curves in \mathbb{P}_1^3 (that is, $\mathbb{P}_1^3 \cap S$) is not generic in the space of the configurations of such curves. So the number of lines might not be $2 \cdot 5^4$, but there are at least $2 \cdot 5^4$ lines, when counted with multiplicity.*

These lines can be classified into two classes. Let

$$\ell_{j_1 j_2} = \mathbb{P}_1^3 \cap \mathbb{P}_{j_1}^3 \cap \mathbb{P}_{j_2}^3, \quad j_1 \neq j_2$$

be the toric subvariety of codimension two in \mathbb{P}_1^3 .

- (1) The line does not intersect $\ell_{j_1 j_2}$ for any j_1, j_2 .
- (2) The line intersects some $\ell_{j_1 j_2}$.

Now we count the number of lines in class (2). The lines in class (2) can be further divided into two classes:

- (2)-I The line intersects two different $\ell_{j_1 j_2}$ s.
- (2)-II The line intersects only one $\ell_{j_1 j_2}$.

Let p be a point in $\ell_{j_1 j_2} \cap S$. Then the number of lines containing p in the above $2 \cdot 5^4$ lines is 25, which is the number of intersection points of the quintic curves C_{j_3}, C_{j_4} , projected to \mathbb{P}^2 from p . There are 30 points like p , so this counts

$$25 \cdot 30$$

lines of class (2). However, this doubly counts the lines of class (2)-I. It is easy to see that the lines in class (2)-I is

$$5^2 \cdot 3,$$

where 3 is the number of pairs of $\ell_{j_1 j_2}$ which does not intersect. Thus, there are

$$25 \cdot 27$$

lines of class (2).

We see that the number of the lines of class (1) is at least,

$$2 \cdot 5^4 - 25 \cdot 27 = 575.$$

This is the number we want, since

$$575 \cdot 5 = 2875$$

is the number of curves in a generic quintic Calabi-Yau hypersurface.

We prove the following, which justifies Katz's count. Originally Katz justified it based on different method [5].

Theorem 9. *For generic f , there are exactly 2875 distinct lines of type (1) and each of them lifts in a unique way. The lines of type (2) do not lift.*

3.2. Proof of the theorem.

3.2.1. Finiteness and transversality. First we prove that the number of lines in \mathbb{P}_1^3 which satisfies the incidence conditions is finite. This is immediate when the incidence conditions (four quintic curves) are generic and the corresponding Schubert calculus gives the expected answer. However, our incidence conditions are in very special position, we prove the finiteness by low-tech argument.

First, we note that the moduli of plane quintic curve is 20 dimensional, so even after fixing 15 points (the position of the intersections with the toric divisors of \mathbb{P}^2), we still have freedom to perturb.

It is clear that for general f , the number of lines of class (2) is finite. A line of class (1) can be described as follows. Recall we write the intersection of \mathbb{P}_1^3 and the singular locus S of \mathfrak{X} by

$$\mathbb{P}_1^3 \cap S = \cup_{j=2}^5 C_j.$$

Fix a point p in C_2 . Consider the projection

$$\pi_p : \mathbb{P}_1^3 \rightarrow \mathbb{P}^2$$

from p . The image of C_3, C_4, C_5 gives three quintic curves in \mathbb{P}^2 . A line through p satisfying the incidence conditions corresponds to the points in the intersection

$$C_3 \cap C_4 \cap C_5.$$

So clearly the number of lines satisfying the incidence conditions is finite.

Moreover, after suitable perturbation, we can assume that for any $p \in C_i$, the triple intersections of the image of quintics by π_p are all transversal in the sense that any two branches out of three intersects transversally.

In this case, the four hypersurfaces in $Gr(4, 2)$, the moduli space of lines in \mathbb{P}_1^3 , given by the incidence conditions intersects transversally, and gives expected $2 \cdot 5^4$ intersections. So the number of lines of class (1) and (2) in \mathbb{P}_1^3 are also as expected, namely, 575 and 675.

3.2.2. Lines intersecting lower dimensional toric strata. In this section, we consider the lifting of the curves of class (2) in Subsection 3.1, and see that these curves do not lift.

We consider a curve of class (2)-II. In this case, the line is contained in the closure of an orbit of a two dimensional subtorus of the big torus acting on \mathbb{P}_1^3 . Let us assume that the curve intersects the line ℓ_{12} (see Subsection 3.1 for the notations).

As before, we consider the local model of the degeneration. In this case, locally around the intersection of the curve and ℓ_{12} (which should be a singular point in the total space of the degeneration), the total space \mathfrak{X} can be written by the equation

$$xyz + tw = 0,$$

where t is the parameter of the degeneration. We assume that the component \mathbb{P}_1^3 corresponds to

$$z = 0, \quad t = 0.$$

The line ℓ_{12} is given by

$$x = y = z = t = 0$$

and the point

$$x = y = z = w = t = 0$$

is the unique singular point of the total space \mathfrak{X} lying on ℓ_{12} , so our curve should intersect this point.

The logarithmic tangent space $\Theta_{\mathfrak{X}}$ is generated by the vectors

$$x\partial_x - y\partial_y, \quad y\partial_y - z\partial_z, \quad w\partial_w - t\partial_t, \quad z\partial_z + w\partial_w.$$

For generic f (the defining polynomial of the quintic hypersurface), the curve is given by

$$x - aw = 0, \quad y - bw = 0, \quad z = t = 0,$$

up to higher order terms. Here the coefficients a, b are not zero. Then the tangent sheaf of the image of the curve should be generated by the vector proportional to

$$x\partial_x + y\partial_y + w\partial_w.$$

However, this vector does not belong to the logarithmic tangent sheaf of \mathfrak{X} , and this means that the curve does not lift to a smoothing of X_0 (in fact, even a local lift does not exist). This can also be seen directly by looking at the ring homomorphisms.

Since the curves of class (2)-I have the same local structure at the intersection with $\ell_{j_1 j_2}$, this argument shows that these curves do not lift, either.

3.2.3. Liftability of lines of class (1). Here, we prove the following. It completes the proof of theorem 9.

Lemma 10. *For generic f , the lines of class (1) lift uniquely.*

Proof. Let $\varphi_0 : \mathbb{P}^1 \rightarrow \mathbb{P}_1^3$ be a line of type (1). Degenerating the family \mathfrak{X} further, we can assume that \mathbb{P}_1^3 torically degenerates so that φ_0 degenerates into a maximally degenerate curve in the sense of [15], Section 5. Let us write the degenerate curve by

$$\psi_0 : C \rightarrow Y_0.$$

Here C is the nodal rational curve with two components, where each component is non-singular. Y_0 is a union of toric varieties obtained by appropriate toric degeneration of \mathbb{P}_1^3 .

Recall that the intersection of the singular locus of the total space \mathfrak{X} with \mathbb{P}_1^3 is the union of four quintic curves. These curves also degenerate to a map to Y_0 , or more specifically, to a map to the union of toric divisors of the components of Y_0 . Let us write by Q the image of this degenerate curve. Recall that such a degeneration is done by toric blow-ups ([15], Proposition 3.9), and the degeneration of the union of these quintic curves will be given by the inverse image of it under these blow-ups. In particular, the original quintic curves are not essentially effected, and we can specify four of the irreducible components of Q , $\{Q_1, Q_2, Q_3, Q_4\}$ which are mapped isomorphically to the four components of the original quintic curves by the blow-down.

Let

$$C = C_1 \cup C_2$$

be the decomposition into irreducible components of the domain of ψ_0 , and

$$\psi_0^i : C_i \rightarrow Y_0^i, \quad i = 1, 2$$

be the restriction of ψ_0 to these components, where Y_0^i is the component of Y_0 to which C_i is mapped. Since ψ_0 is maximally degenerate, for each i , the image of C_i is contained in an orbit of two dimensional subtorus of the torus acting on Y_0^i . Let L_i be the closure of this orbit.

The image of each ψ_0^i intersects the toric divisor of Y_0^i at three points. By construction, two of these points are the points on two components of Q , say $Q^{i,1}, Q^{i,2}$, and the other is not.

Let $Y_0^{i,1}, Y_0^{i,2}$ be the toric divisors of Y_0^i which contain $Q^{i,1}, Q^{i,2}$ respectively. The intersection of L_i with $Y_0^{i,j}$ gives rational curves $\ell^{i,j}$. Note that the each intersection $Q^{i,1} \cap \ell^{i,1}, Q^{i,2} \cap \ell^{i,2}$ contains a point of the image of ψ_0^i .

Case 1: When each $\ell^{i,j}$, $i, j = 1, 2$ is transverse to $Q^{i,j}$ at the images of ψ_0^i .

For curves of type (1), the local structure of the singularity of \mathfrak{X} is isomorphic to the product of a neighborhood of a singular point of the total space of the degeneration of a generic K3 surface, that is, a neighborhood of the origin of the set

$$\{(X, Y, Z, t) \in \mathbb{C}^4 \mid XY + tZ = 0\} \subset \mathbb{C}^3 \times \mathbb{C},$$

and an open subset of \mathbb{C} .

By the calculation in Section 2, when the lines $\ell^{i,j}$, $i, j = 1, 2$ are transverse to the singular locus Q , the normal sheaf \mathcal{N}_i to ψ_0^i is given by

$$\mathcal{O}(-1) \oplus \mathcal{O},$$

where the summand $\mathcal{O}(-1)$ comes from the normal sheaf of ψ_0^i as a map to $L_i \subset \mathfrak{X}$, and the summand \mathcal{O} is the transverse direction to L_i .

Note that the components Y_0^1, Y_0^2 are acted by the same torus, which is the same as the torus acting on \mathbb{P}_1^3 . Let us write this by

$$N \otimes \mathbb{C}^*,$$

where $N \cong \mathbb{Z}^3$. So the spaces L_1, L_2 are the orbit closures of subtori of the same torus, defined by sub-lattices of N of rank two, which we write by N_1, N_2 . It is clear that N_1

and N_2 do not coincide. Then it follows that the normal sheaves \mathcal{N}_i on the components C_i are naturally extended to a sheaf \mathcal{N} on the union C , which is a direct sum

$$\mathcal{N} = \mathcal{L}_1 \oplus \mathcal{L}_2.$$

Here \mathcal{L}_i , $i = 1, 2$ are locally free, and

$$\mathcal{L}_1|_{C_1} \cong \mathcal{O}(-1), \quad \mathcal{L}_1|_{C_2} \cong \mathcal{O},$$

$$\mathcal{L}_2|_{C_1} \cong \mathcal{O}, \quad \mathcal{L}_2|_{C_2} \cong \mathcal{O}(-1).$$

Thus, one sees that the cohomology

$$H^1(C, \mathcal{N})$$

vanishes, so the map ψ_0 is uniquely liftable. Clearly, the lifted curve has the normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, as expected. \square

Case 2: When some $\ell^{i,j}$ is not transverse to $Q^{i,j}$ at the images of ψ_0^i .

By blowing-down some of the toric divisors of $Y_0^{i,j}$, we restore the original situation $Q^{i,j} \subset \mathbb{P}^2$. The line $\ell^{i,j}$ becomes a line through a torus fixed point p of \mathbb{P}^2 . There are finitely many lines through p which is not transverse to $Q^{i,j}$. So after perturbing the incidence conditions, we can assume this case does not happen. This proves the lemma. \square

Remark 11. *We can also argue Clemens' conjecture along this line. I hope I can report some progress in this direction in the near future.*

Remark 12. *The degeneration need not be the ones whose central fiber is a union of projective spaces, but the components of the central fiber can be more general toric variety (in fact, the components of the central fiber need not be toric varieties, though explicit calculations will become difficult in general). We give examples in the cases of a cubic surface and a K3 surface.*

1. Cubic surface.

Consider a degeneration to the union of a projective plane P and a quadratic surface Q . The intersection $P \cap Q$ is a quadratic curve C , and the singular locus S of the total space of degeneration is a set of six points on C . There are ${}_6C_2 = 15$ lines in P which intersects C at S . On the other hand, remembering $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, there are 12 lines in Q which intersects C at S . By our calculation, the normal sheaf is $\mathcal{O}(-1)$ for both types, and all of them lifts, giving 27 lines, as expected.

2. K3 surface.

Consider a degeneration to the union of two quadratic surfaces Q_1, Q_2 . The intersection $Q_1 \cap Q_2$ is a curve C of degree $(2, 2)$ on each of them. The singular locus S of the total space of degeneration is a set of 16 points on C . A line in Q_i intersects C at two points (counted with multiplicity), and the necessary condition for the liftability is that both of these intersections are contained in S . The sufficient condition is given by the vanishing

of the Kuranishi map.

4. DISKS OF MASLOV INDEX ZERO

Using the technique developed in [13], we can apply the calculation here to construct families of holomorphic disks of Maslov index zero in many symplectic manifolds. Such disks play important roles. For example:

- (1) When we calculate the *potential functions* (see [3], Section 4), it is important to count disks of Maslov index two. These disks undergo wall crossing phenomena, and disks of Maslov index zero compose these walls ([1]).
- (2) Same kind of walls appear in the general construction of Calabi-Yau manifolds ([4, 9]) from affine manifolds with singularities.

Theorem 13. *Let $f = f(x_0, \dots, x_n)$ be a general homogeneous polynomial of degree larger than one. Let $V = \{f = 0\} \subset \mathbb{P}^n$ be a reduced projective hypersurface and let $\Omega \in H^2(V, \mathbb{Z})$ be the class determined by the polarization. Then there is a symplectic form ω on V of class Ω such that there is a Lagrangian torus L with respect to ω which bounds a holomorphic disk of Maslov index zero.*

Proof. Let d be the degree of f . Then consider a degeneration

$$a_1 a_2 \cdots a_d + t f = 0$$

of V . Here t is the parameter of the degeneration. We take a_i so that

$$a_i = \begin{cases} x_{i-1}, & i = 1, \dots, n+1. \\ \text{A general linear homogeneous polynomial,} & i > n+1. \end{cases}$$

Let $\mathfrak{X} \subset \mathbb{P}^n \times \mathbb{C}$ be the total space of the degeneration and let

$$\pi : \mathfrak{X} \rightarrow \text{Spec} \mathbb{C}[t]$$

be the projection.

Let S be the singular locus of \mathfrak{X} . Its intersection with a plane

$$\mathbb{P}_{ij}^{n-2} = \mathbb{P}_i^{n-1} \cap \mathbb{P}_j^{n-1}, \quad i \neq j, \quad i, j \leq n+1$$

here $\mathbb{P}_i^{n-1} = \{a_i = 0\}$, is a hypersurface of degree d in \mathbb{P}_{ij}^{n-2} . Let

$$\mathbb{P}_{klm}^{n-3} = \mathbb{P}_k^{n-1} \cap \mathbb{P}_l^{n-1} \cap \mathbb{P}_m^{n-1}, \quad k \neq l \neq m \neq k, \quad 0 \leq k, l, m \leq d$$

be the triple intersection. Let

$$\text{int} \mathbb{P}_{ij}^{n-2} = \mathbb{P}_{ij}^{n-2} \setminus \bigcup_{k \neq i, j} \mathbb{P}_{ijk}^{n-3}$$

be the 'interior' of \mathbb{P}_{ij}^{n-2} .

Let us fix a point $x \in S \cap \text{int} \mathbb{P}_{ij}^{n-2}$ generally. There is a natural structure of a toric variety on \mathbb{P}_i^{n-1} , $i \leq n+1$, so that \mathbb{P}_{ij}^{n-2} , $i, j \leq n+1$, is one of the toric divisors. Then by the description of the disks in toric varieties ([2]), there is a real one dimensional family of Lagrangian tori each of which bounds a unique (up to isomorphism) disk of Maslov index 2 (considered in \mathbb{P}_i^{n-1}) intersecting the point x .

Let L be one of these Lagrangian submanifolds. It is an orbit of the toric action of T^{n-1} on \mathbb{P}_i^{n-1} . The restriction of the Fubini-Study form $\tilde{\omega}_{FS}$ of \mathbb{P}^n to \mathbb{P}_i^{n-1} defines a T^{n-1} -invariant symplectic form ω . We can take a neighborhood U of L in the total space \mathfrak{X} so that the following conditions hold.

- The diagram

$$\begin{array}{ccc} U & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ \text{Spec}\mathbb{C}[t] & \xlongequal{\quad} & \text{Spec}\mathbb{C}[t] \end{array}$$

commutes, where the upper arrow is the inclusion.

- The action of T^{n-1} extends to U over $\text{Spec}\mathbb{C}[t]$.
- The form ω also extends to a T^{n-1} -invariant symplectic form $\tilde{\omega}$ on U .
- For $0 \neq t_0 \in \pi(U) \subset \text{Spec}\mathbb{C}[t]$, the restriction of $\tilde{\omega}$ to $X_{t_0} = \pi^{-1}(t_0)$ can be extended to a symplectic form on X_{t_0} which defines the same class in $H^2(X_{t_0}, \mathbb{Z})$ as the restriction of $\tilde{\omega}_{FS}$.
- On U , the action and the form are (real) analytic with respect to the analytic structure of \mathfrak{X} .

Remark 14. *Note that we do not require that the extension of $\tilde{\omega}$ to X_{t_0} is analytic outside $U \cap X_{t_0}$.*

Then there is a real analytic family \mathcal{L} of Lagrangian tori fibered over $\pi(U)$, which restricts to L on X_0 . In [13], we developed a deformation theory of disks with such Lagrangian boundary condition as \mathcal{L} . Let

$$\varphi_0 : D \rightarrow X_0$$

be the disk with boundary on L and intersecting the point x as in the previous sections. Then by [13], the lifts of φ_0 to the fibers $t \neq 0$ is controlled by the logarithmic normal bundle as in the case of rational curves above, except that we should use *Riemann-Hilbert bundles* ([8]) to take care of the boundary condition.

In our case, the normal bundle of the disk φ_0 , as a map to \mathbb{P}_i^{n-1} , is trivial. However, as in the case of rational curves, we have to take account of the singularity of \mathfrak{X} , which contains the point x .

For simplicity, let us take $i = 0, j = 1$, and use the affine coordinates

$$Y_i = x_i/x_n, \quad i = 0, \dots, n-1.$$

We also set

$$Y_j = a_j/x_n$$

for $j \geq n+1$. Using the coordinates $\{Y_i\}_{i=0}^{n-1}$, the point x is represented as

$$x = (0, 0, b_2, \dots, b_{n-1}),$$

where $b_i \neq 0$ by genericity of x . Also, the linear polynomials $a_i, i > n+1$ are not zero at x . Redefining f by

$$\frac{f}{Y_2 Y_3 \cdots},$$

the defining equation of the degeneration near x is

$$Y_0 Y_1 + t f = 0.$$

Since x belongs to the singular locus,

$$f(x) = 0$$

holds. By genericity of f and x , we can assume the following:

- $\frac{\partial f}{\partial Y_i}(x) \neq 0$, $i = 0, \dots, n-1$.
- Moreover, the arguments of the complex number $\frac{\partial f}{\partial Y_i}(x)$ are all different.

Under this assumption, the function f is pulled back by φ_0 to a holomorphic function on the disk which has a simple zero at $\varphi_0^{-1}(x)$.

Differentiating the equation, we obtain the relation

$$\frac{dY_0}{Y_0} + \frac{dY_1}{Y_1} - \frac{df}{f} - \frac{dt}{t} = 0.$$

Restricting to X_0 , we have

$$\frac{dY_0}{Y_0} + \frac{dY_1}{Y_1} - \frac{df}{f} = 0.$$

We separate the problem into three cases according to $n = 3, 4$ or $n \geq 5$. The case $n \leq 2$ is trivial.

Case 1: $n = 3$. In this case, the singular locus of \mathfrak{X} is zero dimensional, and the argument in Section 2 extends straightforwardly, so that the normal sheaf of φ_0 is the sheaf associated to the Riemann-Hilbert bundle whose doubling is $\mathcal{O}(-1)$ (the normal bundle as a map to \mathbb{P}_0^{n-1} is trivial, and the singularity contributes -1). So the obstruction vanishes and φ_0 lifts.

Case 2: $n = 4$. From here, a difference from the closed curve case appears due to the existence of the boundary condition. Note that by suitable coordinate change as in the previous section, one sees that a neighborhood of $x \in \mathfrak{X}$ is analytically isomorphic to a product P of a neighborhood N of the origin of the set

$$\{(X, Y, Z, t) \in \mathbb{C}^4 \mid XY + tZ = 0\} \subset \mathbb{C}^3 \times \mathbb{C}$$

and an open subset O of \mathbb{C} (this coordinate change does not preserve the toric structure, though. So we cannot use these coordinates when we need to take care of the boundary condition.). In particular, the pull-back of $\Theta_{\mathfrak{X}}$ by φ_0 and the logarithmic normal bundle of φ_0 are locally free. Here, we put a log structure (in the analytic category) on the disk associated to $\varphi_0^{-1}(x)$.

The pull back of $\Theta_{\mathfrak{X}}$ is generated by

$$Y_1 \partial_{Y_1}, \quad \frac{f - Y_1 \frac{\partial f}{\partial Y_1}}{f} Y_0 \partial_{Y_0} - \frac{f - Y_0 \frac{\partial f}{\partial Y_0}}{f} Y_1 \partial_{Y_1},$$

$$Y_1 \partial_{Y_1} + \left(f - Y_1 \frac{\partial f}{\partial Y_1} \right) \cdot \left(Y_2 \frac{\partial f}{\partial Y_2} \right)^{-1} Y_2 \partial_{Y_2}, \quad \frac{1}{Y_2} \frac{\partial f}{\partial Y_3} Y_2 \partial_{Y_2} - \frac{1}{Y_3} \frac{\partial f}{\partial Y_2} Y_3 \partial_{Y_3}$$

around x . Here

$$\frac{dt}{t}(Y_1 \partial_{Y_1}) = 1,$$

and the other vectors are evaluated to zero by $\frac{dt}{t}$. Also, we abused the notation somewhat, namely, the functions $\frac{f-Y_1 \frac{\partial f}{\partial Y_1}}{f}$, $\frac{f-Y_0 \frac{\partial f}{\partial Y_0}}{f}$, etc. are in fact those pulled back to the disk by φ_0 .

A logarithmic tangent vector of the disk is mapped to a constant multiple of $\frac{f-Y_1 \frac{\partial f}{\partial Y_1}}{f} Y_0 \partial_{Y_0} - \frac{f-Y_0 \frac{\partial f}{\partial Y_0}}{f} Y_1 \partial_{Y_1}$, so the relevant normal sheaf is generated by

$$Y_1 \partial_{Y_1} + \left(f - Y_1 \frac{\partial f}{\partial Y_1} \right) \cdot \left(Y_2 \frac{\partial f}{\partial Y_2} \right)^{-1} Y_2 \partial_{Y_2}, \quad \frac{1}{Y_2} \frac{\partial f}{\partial Y_3} Y_2 \partial_{Y_2} - \frac{1}{Y_3} \frac{\partial f}{\partial Y_2} Y_3 \partial_{Y_3}.$$

Now we consider the boundary condition. Since the terms containing Y_1 does not affect the boundary condition, we ignore them. The boundary Lagrangian torus L gives the sheaf of the sections of the Riemann-Hilbert bundle on D

$$(\varphi_0^* \Theta_{\mathbb{P}^3_i}, \mathcal{O}_L),$$

where \mathcal{O}_L is the sheaf of real analytic functions on L , see [13] for more details.

Noting that the image of the disk has constant Y_2 and Y_3 coordinates, the totally real subspace of $\varphi_0^* \Theta_{\mathbb{P}^3_i}|_{\partial D}$ is spanned by

$$\partial_r, \partial_{\theta_2}, \partial_{\theta_3},$$

here ∂_r is the radial direction of the disk (in other words, the real part of $Y_0 \partial_{Y_0}$), and we write

$$Y_2 = r_2 e^{i\theta_2}, \quad Y_3 = r_3 e^{i\theta_3}.$$

Lemma 15. *The normal bundle is a Riemann-Hilbert bundle whose doubling is isomorphic to*

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1)$$

on \mathbb{P}^1 .

Proof of Lemma 15. It suffices to find rational sections which satisfy the boundary condition. First consider $f(\frac{1}{Y_2} \frac{\partial f}{\partial Y_2})^{-1} Y_2 \partial_{Y_2}$ (as mentioned above, we ignore the part containing Y_1). Let S be the affine coordinate on the disk defined by

$$\varphi_0^* Y_0 = S.$$

Recall that the function f is pulled back to a holomorphic function which has simple zero at $\varphi_0^{-1}(x)$. Moreover, Y_2 is pulled back to the nonzero constant b_2 , and $\frac{\partial f}{\partial Y_2}$ is pulled back to a holomorphic function which does not have a zero on the disk (when we take the torus L sufficiently close to the boundary $Y_0 = Y_1 = 0$).

Thus, using the coordinate S , the vector $f(\frac{1}{Y_2} \frac{\partial f}{\partial Y_2})^{-1} Y_2 \partial_{Y_2}$ is written as

$$Sg(S)Y_2 \partial_{Y_2},$$

where $g(S)$ is a holomorphic function which does not have a zero on the disk. Then

$$\frac{i}{Sg(S)} \cdot Sg(S)Y_2 \partial_{Y_2}$$

gives a rational section satisfying the boundary condition. This has a simple pole at x , so defines a subbundle whose double is $\mathcal{O}(-1)$.

Next consider $\frac{1}{Y_2} \frac{\partial f}{\partial Y_3} Y_2 \partial_{Y_2} - \frac{1}{Y_3} \frac{\partial f}{\partial Y_2} Y_3 \partial_{Y_3}$. As above, the coefficients are holomorphic functions which do not have zero, so dividing by $\frac{1}{Y_3} \frac{\partial f}{\partial Y_2}$, we can write it as

$$h(S)Y_2\partial_{Y_2} - Y_3\partial_{Y_3},$$

where $h(S)$ is a holomorphic function which does not have a zero. Then

$$i(h(S)Y_2\partial_{Y_2} - Y_3\partial_{Y_3}) - \frac{ih(S)}{Sg(S)} \cdot Sg(S)Y_2\partial_{Y_2}$$

gives a rational section satisfying the boundary condition. This also has a simple pole at x . Since the normal bundle has rank two, it must be

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1).$$

□

Thus, again the obstruction to lift φ_0 vanishes, so φ_0 lifts. This proves the case 2.

Case 3: $n \geq 5$. In this case, the normal sheaf is generated by

$$Y_1\partial_{Y_1} + \left(f - Y_1 \frac{\partial f}{\partial Y_1}\right) \cdot \left(\frac{1}{Y_2} \frac{\partial f}{\partial Y_2}\right)^{-1} Y_2\partial_{Y_2}, \quad \frac{1}{Y_2} \frac{\partial f}{\partial Y_i} Y_2\partial_{Y_2} - \frac{1}{Y_i} \frac{\partial f}{\partial Y_2} Y_i\partial_{Y_i}, \quad i = 3, \dots, n-1.$$

In this case, its isomorphism class is calculated as follows.

Lemma 16. *The normal bundle is a Riemann-Hilbert bundle whose doubling is isomorphic to*

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus n-4}$$

on \mathbb{P}^1 .

Proof of Lemma 16. The $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ is the same as above. Now we take $\frac{1}{Y_2} \frac{\partial f}{\partial Y_4} Y_2\partial_{Y_2} - \frac{1}{Y_4} \frac{\partial f}{\partial Y_2} Y_4\partial_{Y_4}$. As before, we may take

$$h_1(S)Y_2\partial_{Y_2} - Y_4\partial_{Y_4}$$

instead. Here $h_1(S)$ is a holomorphic function which does not have a zero. Again,

$$i(h_1(S)Y_2\partial_{Y_2} - Y_4\partial_{Y_4}) - \frac{ih_1(S)}{Sg(S)} \cdot Sg(S)Y_2\partial_{Y_2}$$

is a rational section satisfying the boundary condition which have a simple pole at x . Note that the residue of the rational sections $s_1 = \frac{i}{Sg(S)} \cdot Sg(S)Y_2\partial_{Y_2}$, $s_2 = i(h(S)Y_2\partial_{Y_2} - Y_3\partial_{Y_3}) - \frac{ih(S)}{Sg(S)} \cdot Sg(S)Y_2\partial_{Y_2}$ and $s_3 = i(h_1(S)Y_2\partial_{Y_2} - Y_4\partial_{Y_4}) - \frac{ih_1(S)}{Sg(S)} \cdot Sg(S)Y_2\partial_{Y_2}$ are 1, $h(0)$, $h_1(0)$ times the same constant. So there are real numbers r_1, r_2 so that

$$r_1 s_1 + r_2 s_2 + s_3$$

is a holomorphic section satisfying the boundary condition. Since its projection to the summand generated by $Y_4\partial_{Y_4}$ is a constant section, it follows that this section generates the subbundle isomorphic to \mathcal{O} . The same holds for other $i \geq 5$, so the lemma is proved. □

The first cohomology of the Riemann-Hilbert bundle of Lemma 15 vanishes, so again there is no obstruction to lift φ_0 . Thus, φ_0 lifts. This finishes the proof of Theorem 13. □

Note that the zeroth cohomology groups of the normal sheaf, which we write as $(\mathcal{N}, \mathcal{O}_L/\mathcal{O}_D)$ is

$$H^0(D, \partial D; \mathcal{N}, \mathcal{O}_L/\mathcal{O}_D) \cong \mathbb{R}^{n-4}.$$

Let $x \in S \cap \text{int} \mathbb{P}_{ij}^{n-2}$ be a general point as before. Recall S is the singular locus of \mathfrak{X} . \mathbb{P}_{ij}^{n-2} has a natural structure of a toric variety. Let $T \cong (S^1)^{n-2}$ be the Lagrangian torus with respect to this toric structure which contains x . Then $n - 4$ is equal to the dimension of the intersection

$$S \cap T.$$

Since the obstruction vanishes, the disks intersecting general points of $S \cap T$ all lift, and we can also move the fiber T , which makes an $n - 2$ dimensional family of torus. Summarizing, we have the following corollary.

Corollary 17. *In the situation of Theorem 13, there are $2n - 6$ dimensional families of holomorphic disks with Lagrangian boundary condition associated to the singular locus of \mathfrak{X} .*

Since the disk is of real dimension two, the total space of these families is (real) codimension two in X_t . This can be thought of as the geometric realization of the *walls* of [4].

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